JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **200,** 149–161 (1996)
ARTICLE NO. 0196

# On the Convergence Rate in the Uniform Ergodic Theorem*

Takeshi Yoshimoto

*Department of Mathematics, Toyo University, Kawagoe, Saitama 350, Japan*



## INTRODUCTION

Let $X$ be a complex Banach space with norm $\delta_X(\cdot)$ and $B[X, X]$ be the Banach algebra of bounded linear transformations from $X$ to $X$ with norm $\pi_B(\cdot)$.

The purpose of this paper is to investigate the speed of uniform convergence for Cesàro averages of quasi-compact operators in $B[X, X]$. A fundamental result in this direction is the following theorem which generalizes Kryloff and Bogoliouboff's result for Markov operators to the case of quasi-compact operators on general Banach spaces.

THEOREM (Yosida and Kakutani [12]). *If $T \in B[X, X]$ is quasi-compact and power-bounded then there exists a compact projection $\Pi \in B[X, X]$ such that*

$$\pi_B\left[\frac{T + T^2 + \cdots + T^n}{n} - \Pi\right] = O\left(\frac{1}{n}\right).$$

The power-boundedness of quasi-compact operators enables us to say even more. Then the following two theorems constitute the substance of the present paper.

THEOREM 1. *Let $T \in B[X, X]$ be quasi-compact and $\pi_B[T^n] \leq K$ $(n = 0, 1, 2, \dots)$ for some constant $K > 0$. Then for any $\alpha > 0$ there exists a*

* Supported in part by the Special Research Funds from Toyo University.



0022-247X/96 $18.00

*compact projection* $\Pi_\alpha \in B[X, X]$ *such that*

$$\pi_B\left[(A_n^\alpha)^{-1}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}T^k - \Pi_\alpha\right] = O\left(\frac{1}{n^\omega}\right), \qquad \omega = \min(\alpha, 1),$$

*where the* $A_k^\alpha$*'s are the* $(C, \alpha)$*-coefficients of order* $\alpha$.

THEOREM 2. *Let* $\{T(t)\colon t \geq 0\}$ *be a uniformly continuous semigroup in* $B[X, X]$. *Suppose that* (i) $\sup_{t\geq 0}\pi_B[T(t)] = M < \infty$, *and* (ii) $T(1)$ *is quasi-compact*. *Then for any* $\alpha \geq 1$ *there exists a compact projection* $\Pi_\alpha \in B[X, X]$ *such that*

$$\pi_B\left[\frac{\alpha}{u^\alpha}\int_0^u (u-t)^{\alpha-1}T(t)\,dt - \Pi_\alpha\right] = O\left(\frac{1}{u}\right).$$

In Section 1 we prove these theorems. Some related topics are discussed in Section 2. Section 3 is devoted to giving some applications of the results to investigate the asymptotic ergodic behaviors of stationary Markov transition functions.

## 1. PROOFS OF THE THEOREMS

Given a general semigroup $G$ in $B[X, X]$ which contains the identity operator $I$, we denote by $\mathrm{CO}(G)$ the convex hull of $G$ and by $\overline{\mathrm{CO}}(G)$ its closure in the uniform operator topology. The orbit of $x \in X$ under $\mathrm{CO}(G)$ is denoted by $\mathrm{OR}(x)$ and its strong closure is denoted by $\overline{\mathrm{OR}}(x)$. Recall that the $(C, \beta)$-coefficients for a real number $\beta > -1$ are defined by

$$A_0^\beta = 1,$$

$$A_k^\beta = \binom{k+\beta}{k} = \frac{(\beta+1)(\beta+2)\ldots(\beta+k)}{k!}, \qquad k = 1, 2, 3, \ldots .$$

Then it is well known that

$$A_n^\beta = \sum_{k=0}^{n} A_{n-k}^{\beta-1} \sim n^\beta/\Gamma(\beta+1).$$

*Proof of Theorem* 1. For notational convenience we write

$$C_n^\alpha[T] = (A_n^\alpha)^{-1}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}, \qquad n = 0, 1, 2, \ldots .$$

The case $0 < \alpha \leq 1$ has already been proved in [11] in a more general setting. Thus it suffices to prove only the case $\alpha > 1$. First we observe that $\{C_n^\alpha[T]\}$ $(n = 0, 1, 2, \ldots)$ is a system of almost uniformly invariant integrals for the cyclic semigroup $G(T)$, where $G(T) = \{T^n: n = 0, 1, 2, \dots\}$. It is clear that

$$\sup_{n \geq 0} \pi_B\big[C_n^\alpha[T]\big] < \infty, \qquad C_n^\alpha[T] \in B[X, X], \qquad C_n^\alpha[T]x \in \overline{\mathrm{OR}}(x)$$

for every $x \in X$ and all $n = 0, 1, 2, \dots$ . Moreover, we have

$$C_n^\alpha[T](T - I)$$
$$= (A_n^\alpha)^{-1}\left\{-A_n^{\alpha-1} + \sum_{k=0}^{n-1}\big(A_{n-k}^{\alpha-1} - A_{n-k-1}^{\alpha-1}\big)T^{k+1} + T^{n+1}\right\}$$

so that

$$\begin{aligned}\pi_B\big[C_n^\alpha[T](T - I)\big] &\leq (A_n^\alpha)^{-1}\big\{A_n^{\alpha-1} + K\big(A_n^{\alpha-1} + 2\big)\big\}\\ &\leq n^{-1}M_d(\alpha), \qquad n = 1, 2, 3, \dots,\end{aligned}$$

where $M_d(\alpha)$ is some positive constant depending only on $\alpha$. From this it follows that $\pi_B[C_n^\alpha[T](T - I)] \to 0$ as $n \to \infty$. We can therefore apply Eberlein's uniform ergodic theorem [4] to assert that there exists a compact projection $\Pi_\alpha \in B[X, X]$ with $T\Pi_\alpha = \Pi_\alpha T = \Pi_\alpha$ and $\Pi_\alpha C_n^\alpha[T] = C_n^\alpha[T]\Pi_\alpha = \Pi_\alpha$ $(n = 0, 1, 2, \dots)$, such that $\Pi_B[C_n^\alpha[T] - \Pi_\alpha] \to 0$ as $n \to \infty$. Thus by [9, Theorem 4.2] we have the decomposition $X = N(I - T) \oplus (I - T)X$ and $(I - T)X$ is closed, where $N(I - T)$ stands for the null space of the operator $I - T$. Using the open mapping theorem, there exists a constant $\Gamma > 0$ such that to each $x \in X$ corresponds a $u \in X$ with

$$x - \Pi_\alpha x = (I - T)u, \qquad \delta_X(u) \leq \Gamma \cdot \delta_X(x - \Pi_\alpha x).$$

Hence we have

$$\begin{aligned}(C_n^\alpha[T] - \Pi_\alpha)x &= (C_n^\alpha[T] - \Pi_\alpha)(I - \Pi_\alpha)x\\ &= (C_n^\alpha[T] - \Pi_\alpha)(I - T)u = C_n^\alpha[T](I - T)u\end{aligned}$$

and so, for all $n = 1, 2, 3, \dots$

$$\begin{aligned}\delta_X\big((C_n^\alpha[T] - \Pi_\alpha)x\big) &= \delta_X\big(C_n^\alpha[T](I - T)u\big)\\ &\leq \pi_B\big[C_n^\alpha[T](T - I)\big]\delta_X(u)\\ &\leq n^{-1}\{\Gamma \cdot M_d(\alpha) \cdot \pi_B[I - \Pi_\alpha]\}\delta_X(x)\end{aligned}$$

as was to be shown, and the proof is complete.

*Proof of Theorem* 2. For brevity we write $G = \{T(t): t \geq 0\}$ and

$$C_u^\alpha[G] = \frac{\alpha}{u^\alpha}\int_0^u (u - t)^{\alpha-1}T(t)\,dt, \qquad u > 0.$$

Since we have

$$C_u^\alpha[G] = uo - \lim_{n\to\infty}\left\{\frac{\alpha}{u^\alpha}\int_{1/n}^{u-1/n}(u-t)^{\alpha-1}T(t)\,dt \Big/ \frac{\alpha}{u^\alpha}\int_{1/n}^{u-1/n}(u-t)^{\alpha-1}\,dt\right\},$$

where the integrals can be approximated by suitable Riemann sums, we see that

$$\sup_{u>0}\pi_B\left[C_u^\alpha[G]\right] < \infty, \qquad C_u^\alpha[G] \in \overline{\mathrm{CO}}(G), \qquad C_u^\alpha[G]x \in \overline{\mathrm{OR}}(x)$$

for every $x \in X$ and all $u > 0$. Moreover, for $0 < t < u$

$$\begin{aligned} C_u^\alpha[G](T(t) - I) = \frac{\alpha}{u^\alpha}\Bigg\{&\int_u^{u+t}(u+t-s)^{\alpha-1}T(s)\,ds \\ &- \int_0^t (u-s)^{\alpha-1}T(s)\,ds \\ &+ \int_t^u\left[(u+t-s)^{\alpha-1} - (u-s)^{\alpha-1}\right]T(s)\,ds\Bigg\} \end{aligned}$$

and hence

$$\begin{aligned} \pi_B\left[C_u^\alpha[G](T(t) - I)\right] &\le \frac{\alpha M}{u^\alpha}\left[u^\alpha - (u-t)^\alpha + tu^{\alpha-1}\right] \\ &= \alpha M\left[\frac{t}{u} + \left\{1 - \left(1 - \frac{t}{u}\right)^\alpha\right\}\right] \to 0 \qquad \text{as } u \to \infty. \end{aligned}$$

This shows that $\{C_u^\alpha[G]\}$ $(u > 0)$ is a system of almost uniformly invariant integrals for $G$. Thus, making use of the Eberlein theorem, there exists a compact projection $\Pi_\alpha \in B[X, X]$ such that $\pi_B[C_u^\alpha[G] - \Pi_\alpha] \to 0$ as $u \to \infty$. Now we may and shall assume that $u \ge 1$. Let $n = [u]$ (the integer part of $u$) so that $u = n + r$ with $0 \le r < 1$. Then

$$C_u^1[G] = nu^{-1}C_n^1[T(1)]C_1^1[G] + ru^{-1}T(n)C_r^1[G].$$

This gives $\Pi_1 = EC_1^1[G]$, where $E = uo\text{-}\lim_{n\to\infty} C_n^1[T(1)]$ and $\pi_B$ $[C_n^1[T(1)] - E] = O(n^{-1})$ which follows from the Yosida–Kakutani theorem. Therefore

$$C_u^1[G] - \Pi_1 = nu^{-1}\{C_n^1[T(1)] - E\}C_1^1[G] \\ +(nu^{-1} - 1)EC_1^1[G] + ru^{-1}T(n)C_r^1[G]$$

so that

$$\pi_B\left[C_u^1[G] - \Pi_1\right] \leq K(u^{-1}), \qquad u \geq 1$$

for some constant $K > 0$ independent of $u$. On the other hand, if $\alpha > 1$ then

$$C_u^\alpha[G] = \frac{\Gamma(\alpha + 1)}{\Gamma(2)\Gamma(\alpha - 1)} u^{-\alpha}\int_0^u t(u - t)^{\alpha - 2} C_t^1[G]\,dt,$$

$$u^{-\alpha}\int_0^u t(u - t)^{\alpha - 2}\,dt = \frac{\Gamma(2)\Gamma(\alpha - 1)}{\Gamma(\alpha + 1)}.$$

Hence from the above estimates we have

$$\begin{aligned}\pi_B\left[C_u^\alpha[G] - \Pi_1\right] &\leq \frac{\Gamma(\alpha + 1)}{\Gamma(2)\Gamma(\alpha - 1)} u^{-\alpha}\int_0^u t(u - t)^{\alpha - 2} \\ &\qquad \pi_B\left[C_t^1[G] - \Pi_1\right] dt \\ &\leq \frac{\Gamma(\alpha + 1)}{(\alpha - 1)\Gamma(2)\Gamma(\alpha - 1)} \\ &\qquad \times\left\{K + \sup_{0 \leq t \leq 1} \pi_B\left[C_t^1[G] - \Pi_1\right]\right\} \cdot u^{-1} \\ &= u^{-1}M_c(\alpha), \qquad u \geq 1.\end{aligned}$$

This implies that $\Pi_\alpha = \Pi_1$ for all $\alpha \geq 1$ and the proof is complete.

*Remark.* It is worthwhile to notice that the following statements can be proved by using the same method as above.

(1) If $T \in B[X, X]$ is quasi-compact and $T^n/n^\xi \to 0$ $(n \to \infty)$ in the weak operator topology for some $\xi$ with $0 < \xi < 1$, then for any $\alpha \geq 1$ there exists a compact projection $\Pi_\alpha^{(d)} \in B[X, X]$ such that

$$\pi_B\left[C_n^\alpha[T] - \Pi_\alpha^{(d)}\right] = O(n^{-(1 - \xi)}).$$

(2) If $G = \{T(t)\colon t \geq 0\}$ is a uniformly continuous semigroup with $T(1)$ quasi-compact and $T(t)/t^\xi \to 0$ $(t \to \infty)$ in the uniform operator

topology for some $\xi$ with $0 < \xi < 1$, then for any $\alpha \geq 1$ there exists a compact projection $\Pi_\alpha^{(c)} \in B[X, X]$ such that

$$\pi_B\left[C_u^\alpha[G] - \Pi_\alpha^{(c)}\right] = O(u^{-(1-\xi)}).$$

## 2. FURTHER DISCUSSIONS

Let $G = \{T(t): t \geq 0\} \subset B[X, X]$ be a uniformly continuous semigroup of type $\omega_0$:

$$\omega_0 = \inf_{t>0} t^{-1} \log \pi_B[T(t)] = \lim_{t\to\infty} t^{-1} \log \pi_B[T(t)].$$

The Abel mean for $G$ is defined by

$$J_\lambda[G] = \int_0^\infty e^{-\lambda t} T(t)\, dt, \qquad \lambda > \omega_0.$$

Then the infinitesimal generator $A$ of $G$ exists and is bounded, and the resolvent $R(\lambda; A)$ of $A$ is identical with $J_\lambda[G]$ for $\lambda > \omega_0$. Furthermore

$$\lambda J_\lambda[G] = \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty t^\alpha e^{-\lambda t} C_t^\alpha[G]\, dt, \qquad \alpha > 0,\ \lambda > \max(0, \omega_0).$$

THEOREM 3. *Suppose that for each $t > 0$, $T(t)$ is compact, with $T(t)/t \to 0$ as $t \to \infty$ in the uniform operator topology. Then*:

(1) *For each $\lambda > 0$ there exists a compact projection $E_\lambda \in B[X, X]$ such that*

$$\pi_B\left[C_n^1[\lambda J_\lambda[G]] - E_\lambda\right] = O\left(\frac{1}{n}\right),$$

$$\pi_B\left[C_n^1[\lambda R(\lambda; A)] - E_\lambda\right] = O\left(\frac{1}{n}\right).$$

(2) *There exists a compact projection $\Pi \in B[X, X]$ such that*

$$\pi_B\left[C_u^\alpha[G] - \Pi\right] \to 0 \qquad \text{as } u \to \infty\ (\alpha \geq 1),$$

$$\pi_B\left[\lambda J_\lambda[G] - \Pi\right] \to 0 \qquad \text{as } \lambda \to 0+.$$

(3) *$X = N(I - \lambda J_\lambda[G]) \oplus (I - \lambda J_\lambda[G])X$ and $(I - \lambda J_\lambda[G])X$ is closed for each $\lambda > 0$.*

(4) *$X = N(A) \oplus R(A)$ and $R(A)$ (the range of $A$) is closed.*

(5) *The point 1 is a simple pole of $\lambda J_\lambda[G]$ for each $\lambda > 0$.*

(6) *The point 0 is a simple pole of $R(\lambda; A)$ with residue $\Pi$ in* (2).

*Proof.* If $t$ is sufficiently large then by assumption, $\pi_B[T(t)] \le t$ and so

$$\omega_0 = \lim_{t\to\infty} t^{-1} \log \pi_B\big[T(t)\big] \le \lim_{t\to\infty} t^{-1} \log t = 0.$$

Let $\lambda > 0$ be fixed. We observe that $J_\lambda[G]$ is compact. To show this we write

$$J_\lambda^{(n)}[G] = \frac{1}{n}\sum_{k=1}^{\infty} e^{-\lambda k/n} T\left(\frac{k}{n}\right), \qquad n = = 1, 2, 3, \dots,$$

$$J_{\lambda,N}^{(n)}[G] = \frac{1}{n}\sum_{k=1}^{N} e^{-\lambda k/n} T\left(\frac{k}{n}\right), \qquad n = 1, 2, 3, \dots; N = 1, 2, 3, \dots .$$

Fix $\varepsilon > 0$ arbitrarily. Then there is a number $t_0 = t_0(\varepsilon) > 0$ and a constant $M > 0$ such that

$$\pi_B\big[T(t)\big] \le \varepsilon \cdot t, \qquad t > t_0$$

$$\pi_B\big[T(t)\big] \le M, \qquad 0 \le t \le t_0.$$

Put $N_0(\varepsilon, n) = [nt_0] + 1$ and $N_0 = N_0(\varepsilon, n)$. For each fixed $n$ one gets

$$\begin{aligned} &\pi_B\Big[J_\lambda^{(n)}[G] - J_{\lambda,N}^{(n)}[G]\Big] \\ &\quad \le \varepsilon \cdot \sum_{k=N}^{\infty} e^{-\lambda k/n} \cdot \frac{k}{n} \\ &\quad \le \left\{\frac{e^{-\lambda/n}}{n(1 - e^{-\lambda/n})} + \frac{ne^{-2\lambda/n}}{n^2(1 - e^{-\lambda/n})^2}\right\} \cdot \varepsilon, \qquad N \ge N_0. \end{aligned}$$

Therefore $J_\lambda^{(n)}[G]$ is compact since $J_{\lambda,N}^{(n)}[G]$ is compact for all $N = 1, 2, 3, \dots$ . Furthermore

$$\begin{aligned} J_\lambda[G] - J_\lambda^{(n)}[G] &= \sum_{k=1}^{\infty}\left\{\int_{k/n}^{(k+1)/n} e^{-\lambda t} T(t)\,dt - \frac{1}{n} e^{-\lambda k/n} T\left(\frac{k}{n}\right)\right\} \\ &\quad + \int_0^{1/n} e^{-\lambda t} T(t)\,dt \\ &= \sum_{k=1}^{\infty} e^{-\lambda k/n} T\left(\frac{k}{n}\right)\left\{\int_0^{1/n} (e^{-\lambda t} - 1) T(t)\,dt\right. \\ &\qquad\qquad \left. + \int_0^{1/n} (T(t) - I)\,dt\right\} \\ &\quad + \int_0^{1/n} e^{-\lambda t} T(t)\,dt. \end{aligned}$$

However, we have for $n > 1/t_0$ the estimates

$$\begin{aligned}
&\pi_B\left[\frac{1}{n}\sum_{k=1}^{\infty} e^{-\lambda k/n} T\left(\frac{k}{n}\right)\right] \\
&\qquad \le \frac{1}{n}\left\{\sum_{k=1}^{N_0-1} e^{-\lambda k/n}\pi_B\left[T\left(\frac{k}{n}\right)\right] + \sum_{k=N_0}^{\infty} e^{-\lambda k/n}\pi_B\left[T\left(\frac{k}{n}\right)\right]\right\} \\
&\qquad \le \frac{1}{n}\left\{M\cdot\sum_{k=1}^{\infty} e^{-\lambda k/n} + \varepsilon\cdot\sum_{k=1}^{\infty} e^{-\lambda k/n}\cdot\frac{k}{n}\right\} \\
&\qquad = \frac{Me^{-\lambda/n}}{n(1-e^{-\lambda/n})} + \frac{\varepsilon}{n}\left\{\frac{e^{-\lambda/n}}{n(1-e^{-\lambda/n})} + \frac{ne^{-2\lambda/n}}{n^2(1-e^{-\lambda/n})^2}\right\}
\end{aligned}$$

and

$$\pi_B\left[n\int_0^{1/n}(e^{-\lambda t}-1)T(t)\,dt\right] \le nM\int_0^{1/n}(1-e^{-\lambda t})\,dt.$$

Hence $\pi_B[J_\lambda^{(n)}[G] - J_\lambda[G]] \to 0$ as $n \to \infty$ and $J_\lambda[G]$ is compact. Using the equation

$$(J_\lambda[G])^n x = \frac{1}{(n-1)!}\int_0^\infty e^{-\lambda t}t^{n-1}T(t)x\,dt$$

we see by a direct computation that $\pi_B[(\lambda J_\lambda[G])^n/n] \to 0$ as $n \to \infty$ [8, Lemma 1]. Thus (1) follows at once from [7, Theorem 2.1] or [11, Theorem 3.2]. Noting that $\pi_B[C_n^1[J_1[G]] - E_1] \to 0$ as $n \to \infty$ taking $\lambda = 1$ in (1), we see by [8, Theorem] that there exists a compact projection $\Pi \in B[X, X]$ such that $\pi_B[C_u^1[G] - \Pi] \to 0$ as $u \to \infty$. From this we have for $\alpha > 1$

$$\begin{aligned}
&\pi_B[C_u^\alpha[G] - \Pi] \\
&\qquad \le \frac{\Gamma(\alpha+1)}{\Gamma(2)\Gamma(\alpha-1)}u^{-\alpha}\int_0^u t(u-t)^{\alpha-2}\pi_B\left[C_t^1[G]-\Pi\right]dt \to 0 \\
&\qquad\qquad \text{as } u \to \infty
\end{aligned}$$

as before, and (2) follows. Part (1) and the Dunford uniform ergodic theorem [2, Theorem 3.16] give (3) at once. Parts (4) and (6) result from (2) and the Hille−Phillips theorem [6, Theorem 18.8.4]. Part (5) is obtained from [3, Chap. VIII, Theorem 8.3] because of the compactness of $J_\lambda[G]$. Hence the proof of the theorem has been completed.

COROLLARY 1. *Let $G = \{T(t): t \geq 0\} \subset B[X, X]$ be a uniformly bounded and uniformly continuous semigroup. Suppose that $T(1)$ is quasi-compact. Then there exists a compact projection $\Pi \in B[X, X]$ such that*

$$\pi_B[\lambda J_\lambda[G] - \Pi] = O(\lambda) \qquad (\lambda \to 0+).$$

THEOREM 4. *Let $\alpha \geq 1$ and let $\{T(t): t \geq 0\} \subset B[X, X]$ be a uniformly bounded strongly continuous semigroup with the infinitesimal generator $A$. Then for each $x \in X$ of the form $x = y + z$ with $y \in N(A)$ and $z \in R(A)$,*

$$\delta_X(C_u^\alpha[G]x - y) = O\left(\frac{1}{u}\right).$$

*Proof.* Let $z = Aw$. Then since $T(0) = I$ we have

$$\begin{aligned} C_u^1[G]x - y &= u^{-1}\int_0^u \frac{d}{ds}T(s)w\,ds \\ &= u^{-1}\{T(u) - I\}w, \qquad u > 0 \end{aligned}$$

so that

$$\delta_X\left(C_u^1[G]x - y\right) \leq u^{-1}\cdot\delta_X(w)\cdot\left(2\sup_{u\geq 0}\pi_B[T(u)]\right), \qquad u > 0.$$

Therefore we have for $\alpha > 1$

$$\begin{aligned} &\delta_X(C_u^\alpha[G]x - y) \\ &\quad \leq \frac{\Gamma(\alpha+1)}{\Gamma(2)\Gamma(\alpha-1)}u^{-\alpha}\int_0^u t(u-t)^{\alpha-2}\delta_X\left(C_t^1[G]x - y\right)dt \\ &\quad \leq u^{-1}\cdot\frac{\Gamma(\alpha+1)\delta_X(w)}{(\alpha-1)\Gamma(2)\Gamma(\alpha-1)}\left(2\sup_{u\geq 0}\pi_B[T(u)]\right), \qquad u > 0 \end{aligned}$$

proving the theorem.

Theorem 4 is the $(C, \alpha)$ $(\alpha \geq 1)$-generalization of Goldstein, Radin, and Showalter's result [5, Theorem 1] concerning the speed of pointwise strong convergence of the $(C, 1)$-averages on $N(A) \oplus R(A)$.

COROLLARY 2. *Let $G = \{T(t): t \geq 0\} \subset B[X, X]$ be a uniformly bounded and uniformly continuous semigroup with the infinitesimal generator $A$. Then for each $x \in X$ of the form $x = y + z$ with $y \in N(A)$ and $z \in R(A)$*

$$\delta_X(\lambda J_\lambda[G]x - y) = O(\lambda) \qquad (\lambda \to 0+).$$

## 3. APPLICATIONS

Let $\Omega = [0, 1]$ and $\mathscr{B}[\Omega]$ = the class of all Borel subsets of $\Omega$. Let $X$ be the complex Banach space of all complex valued $\sigma$-additive set functions $x(\cdot)$ defined for all $A \in \mathscr{B}[\Omega]$ with norm $\delta_X(x(\cdot))$ = total variation of $|x(A)|$ on $\Omega$. We consider a stationary Markov transition function $P(t, \xi, A)$ ($t \geq 0$, $\xi \in \Omega$, $A \in \mathscr{B}[\Omega]$) which satisfies the conditions:

(1) $P(t, \cdot\,, A)$ is a Borel function of $\xi$ for fixed $t, A$;

(2) $P(t, \xi, \cdot\,)$ is a probability measure in $A$ for fixed $t, \xi$;

(3) $P(0, \xi, A) = e_A(\xi)$, $e_A(\xi)$ denotes the indicator function of $A$;

(4) $P(s+t, \xi, A) = \int_\Omega P(t, \eta, A) P(s, \xi, d\eta)$ (the Chapman–Kolmogorov equation).

We define with the kernel $P(t, \xi, A)$

$$T(t)x(A) = \int_\Omega x(d\xi) P(t, \xi, A), \qquad t \geq 0,\ A \in \mathscr{B}[\Omega],\ x \in X.$$

Then $\{T(t)\colon t \geq 0\}$ is a semigroup of positive linear operators in $B[X, X]$ with $T(0) = I$ and $\pi_B[T(t)] \leq 1$ ($t \geq 0$). In addition, we impose the following condition upon the function $P(t, \xi, A)$:

(5) $\lim_{t \to 0+} P(t, \xi, A) = e_A(\xi)$ uniformly in $\xi, A$.

Given any $\varepsilon > 0$ there exists by condition (5) a number $\delta = \delta(\varepsilon) > 0$ such that

$$|P(t, \xi, A) - e_A(\xi)| < \varepsilon, \qquad 0 \leq t < \delta$$

uniformly in $\xi$ and $A$. Thus we have for any $x \in X$

$$\delta_X([T(t) - I]x(\cdot))\varepsilon \cdot \delta_X(x(\cdot))$$

from which it follows that $\{T(t)\colon t \geq 0\}$ is uniformly continuous.

THEOREM 5. *Let $\alpha \geq 1$ and suppose that $T(1)$ is quasi-compact. Then there exists a transition probability function $Q^{(\alpha)}(t, \xi, A)$ ($t \geq 0$, $\xi \in \Omega$, $A \in \mathscr{B}[\Omega]$) such that*

(i)
$$\operatorname*{l.u.b.}_{\xi \in \Omega,\, A \in \mathscr{B}[\Omega]} \left| \frac{\alpha}{u^\alpha} \int_0^u (u-t)^{\alpha-1} P(t+s, \xi, A)\, dt - Q^{(\alpha)}(s, \xi, A) \right| = O\left(\frac{1}{u}\right), \qquad s \geq 0;$$

(ii)
$$\int_\Omega P(t, \xi, d\eta) Q^{(\alpha)}(s, \eta, A) = Q^{(\alpha)}(t+s, \xi, A), \qquad s, t \geq 0.$$

*Proof.* Apply Theorem 2 to the semigroup $G = \{T(t): t \geq 0\}$ to obtain a compact projection $\Pi_\alpha \in B[X, X]$ such that $\pi_B[C_u^\alpha[G] - \Pi_\alpha] = O(u^{-1})$. Using $\Pi_\alpha$ we define

$$Q^{(\alpha)}(t, \xi, A) = \Pi_\alpha P(t, \xi, A), \qquad t \geq 0, \xi \in \Omega, A \in \mathscr{B}[\Omega].$$

Then we have

$$\begin{aligned} &\underset{\xi \in \Omega, A \in \mathscr{B}[\Omega]}{\text{l.u.b.}} \left| \frac{\alpha}{u^\alpha} \int_0^u (u-t)^{\alpha-1} P(t+s, \xi, A)\, dt - Q^{(\alpha)}(s, \xi, A) \right| \\ &\qquad \leq \pi_B[C_u^\alpha[G] - \Pi_\alpha] \cdot \underset{\xi \in \Omega}{\text{l.u.b.}}\, \delta_X(P(s, \xi, \cdot)), \end{aligned}$$

and (i) follows. Clearly $Q^{(\alpha)}(t, \cdot\,, A)$ is a Borel function of $\xi$ for fixed $t$, $A$, and $Q^{(\alpha)}(t, \xi, \cdot\,)$ is a probability measure in $A$ for fixed $t$, $\xi$. As for (ii) we have by (i) and (4)

$$\begin{aligned} &Q^{(\alpha)}(t+s, \xi, A) \\ &\qquad = \lim_{u \to \infty} \frac{\alpha}{u^\alpha} \int_0^u (u-\omega)^{\alpha-1} P(\omega + t + s, \xi, A)\, d\omega \\ &\qquad = \lim_{u \to \infty} \frac{\alpha}{u^\alpha} \int_0^u (u-\omega)^{\alpha-1}\, d\omega \left[ \int_\Omega P(\omega + s, \eta, A) P(t, \xi, d\eta) \right] \\ &\qquad = \int_\Omega P(t, \xi, d\eta) \left[ \lim_{u \to \infty} \frac{\alpha}{u^\alpha} \int_0^u (u-\omega)^{\alpha-1} P(\omega + s, \eta, A)\, d\omega \right] \\ &\qquad = \int_\Omega P(t, \xi, d\eta) Q^{(\alpha)}(s, \eta, A), \end{aligned}$$

and the theorem is proved.

It should be noticed that if $P(1, \xi, A)$ satisfies the Doeblin (D)-condition then the operator $T(1)$ is quasi-compact by the Yosida−Kakutani theorem [12, Theorem 13].

Let $W$ be the infinitesimal generator of $\{T(t): t \geq 0\}$. From Theorem 4 we have

THEOREM 6. *Let* $\alpha \geq 1$. *If* $P(t_0, \xi_0, \cdot\,) \in N(W) \oplus R(W)$ *for some* $t_0$ *and* $\xi_0$ *then*

$$\delta_X\left( \frac{\alpha}{u^\alpha} \int_0^u (u-t)^{\alpha-1} P(t + t_0, \xi_0, \cdot)\, dt - P_N(t_0, \xi_0, \cdot) \right) = O\left(\frac{1}{u}\right),$$

*where* $P(t_0, \xi_0, \cdot\,) = P_N(t_0, \xi_0, \cdot\,) + P_R(t_0, \xi_0, \cdot\,)$, $P_N(t_0, \xi_0, \cdot\,) \in N(W)$, $P_R(t_0, \xi_0, \cdot\,) \in R(W)$.

*Remark.* For each $\xi \in \Omega$, $P(0, \xi, \cdot) \in D(W)$ (the domain of $W$) and we have

$$-WP(0, \xi, A) = \lim_{t \to 0+} \frac{e_A(\xi) - P(t, \xi, A)}{t}.$$

Indeed, this follows from [1, Chap. VI, Theorems 2.2 and 2.3]. With $WP(0, \xi, A)$ define

$$q(\xi) = -WP(0, \xi, \{\xi\}), \qquad \xi \in \Omega,$$

$$q(\xi, A) = WP(0, \xi, A), \qquad \xi \in \Omega - A,\ A \in \mathscr{B}[\Omega].$$

Then $q(\xi, A) \leq q(\xi)$ for all $\xi \in \Omega$ and all $A \in \mathscr{B}[\Omega]$ with $A \subset \Omega - \{\xi\}$. More precisely, the pair $[q(\cdot), q(\cdot, \cdot)]$ constitutes a standard pair of $q$-functions in the sense of Doob (see [1, Chap. VI, Theorem 2.4]). Then $P(t, \xi, \cdot) \in D(W)$ for each $t > 0$ and each $\xi \in \Omega$, and

$$\begin{aligned} WP(t, \xi, A) = -q(\xi)P(t, \xi, A) \\ + \int_{\Omega - \{\xi\}} P(t, \eta, A) q(\xi, d\eta), \qquad A \in \mathscr{B}[\Omega]. \end{aligned}$$

To see this, we need the following representation (see [1, p. 269])

$$\begin{aligned} P(t, \xi, A) = \int_0^t \left\{ \int_{\Omega - \{\xi\}} e^{-q(\xi)s} P(t - s, \eta, A) q(\xi, d\eta) \right\} ds \\ + e^{-q(\xi)t} e_A(\xi). \end{aligned}$$

Thus, keeping $\xi$ fixed and differentiating with respect to $t$ we have

$$\frac{\partial P(t, \xi, A)}{\partial t} = -q(\xi)P(t, \xi, A) + \int_{\Omega - \{\xi\}} P(t, \eta, A) q(\xi, d\eta)$$

for $t > 0$, so that $P(t, \xi, \cdot) \in D(W)$ and $WP(t, \xi, A) = \partial P(t, \xi, A)/\partial t$.

Next let $J_\alpha = so\text{-}\lim_{u \to 0+} C_u^\alpha[G]$ for any real $\alpha > 0$ and $G = \{T(t) : t \geq 0\}$. From the uniform continuity of $G$ it follows that $P(s, \xi, \cdot) \in D(J_\alpha)$ for any $s \geq 0$ and all $\xi \in \Omega$ and that

$$\lim_{u \to 0+} \frac{\alpha}{u^\alpha} \int_0^u (u - t)^{\alpha - 1} P(t + s, \xi, A)\, dt = J_\alpha P(s, \xi, A)$$

uniformly in $\xi, A$. If $P(s_0, \xi, A)$ satisfies the Doeblin (D)-condition for some $s_0 > 0$ then $T(s_0)$ is quasi-compact. In this case, the set $\{C_u^\alpha[G]P(s_0, \xi, A) : u > 0\}$ contains a weakly convergent subsequence $\{C_{u_j}^\alpha[G]P(s_0, \xi, A)\}$ $(j = 1, 2, 3, \ldots)$ with $u_j \to 0+$ as $j \to \infty$ and by [10, Theorem 1] we have $D(J_\alpha) = X$, where $J_\alpha^2 = J_\alpha$ and $J_\alpha T(t) = T(t)J_\alpha = T(t)$ for all $t > 0$.

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